§ Homogeneous equation with constant coefficient:

$$a_n y^{(n)} + \cdots + a_0 y = 0 \quad , \quad a_n \neq 0, \ldots, a_0 \in \mathbb{R}$$

Def: Char equation:

$$a_n r^n + \cdots + a_0 = 0$$

Char polynomial:

$$f(r) = a_n r^n + \cdots + a_0$$

Thm: (Fundamental theorem of algebra)

$f(r) = a_n(r - r_1) \cdots (r - r_n)$ for some $r_1, \ldots, r_n \in \mathbb{C}$!

Pf: Three ways to prove it

1) use Galois theory
2) use complex analysis
3) use topology

we skip it!

Def: Let r_0 be root of $f(r) = 0$, ($\in \mathbb{C}$)
if we can write $f(r) = S_{n-k}(r)\,(r - r_0)^k$
for $(n-k)$ order polynomial s.t. $S_{n-k}(r_0) \neq 0$
We said multiplicity of r_0 is k.

Rk: • In general we can write

$$f(r) = a_n (r-r_1)^{k_1} \cdots\cdots (r-r_s)^{k_s}$$ for distinct

roots $r_1, \dots, r_s$ $\in \mathbb{C}$ with multiplicity $k_1, \dots, k_s$

• If we have $f(r) = a_n r^n + \cdots + a_0$, $a_n \in \mathbb{R}$, $a_0 \in \mathbb{R}$, and

if $r_1 \in \mathbb{C}$, with $r_1 \neq \overline{r_1}$ and r_1 has multiplicity k_1

$$\Longrightarrow \quad f(r) = S_{n-k_1}(r)\,(r-r_1)^{k_1}$$

and taking conjugate:

$$f(r) = \overline{f(r)} = \overline{S_{n-k_1}(r)}\,(r-\overline{r_1})^{k_1}$$

i.e. $\overline{r_1}$ is also a root of multiplicity k_1

In general:

$$f(r) = a_n \underbrace{(r-r_1)^{k_1} \cdots (r-r_s)^{k_2}}_{\text{real roots.}}$$

$$\underbrace{(r-\lambda_1)^{l_1}(r-\overline{\lambda_1})^{l_1} \cdots (r-\lambda_r)^{l_r}(r-\overline{\lambda_r})^{l_r}}_{\text{Complex roots.}}$$

Eg. 1: • if $f(r)=0$ has $r_1 \neq r_2 \ldots \neq r_n$ distinct real roots

• then $y_1(t) = e^{r_1 t}, \ldots\ldots, y_n(t) = e^{r_n t}$ is a fundamental set of solution.

The Wronskian:

$$W(y_1, \ldots, y_n)(0) = \det \begin{pmatrix} 1 & \cdots & 1 \\ r_1 & & r_n \\ \vdots & \ddots & \vdots \\ r_1^{n-1} & \cdots & r_n^{n-1} \end{pmatrix}$$

Ex:

$$\neq 0$$

Eg. 2: if $f(r) = a_n \underbrace{(r-r_1)\ldots(r-r_s)}_{\text{simple root}} \underbrace{(r-r_0)^{n-s}}_{\text{mult.} = n-s.}$

• We have $y_1(t) = e^{r_1 t}, \ldots\ldots, y_s(t) = e^{r_s t}$ are solution to the equation.

• $y_{s+1}(t) = e^{r_0 t}$ is also a solution.

Q: How to find other solutions?

Guess: $y_{s+2}(t) = te^{r_0 t}, \ldots\ldots, y_n(t) = t^{n-s-1} e^{r_0 t}$

Check: $y_{s+1}(t), \ldots\ldots, y_n(t)$ are solution to the equation

We let $f(\frac{d}{dt}) = a_n \frac{d^n}{dt^n} + \ldots + a_1 \frac{d}{dt} + a_0$

and $f(\frac{d}{dt})(y) = a_n \frac{d^n}{dt^n}(y) + \ldots + a_1 \frac{d}{dt}(y) + a_0$

On the other hand:

$$f(\frac{d}{dt}) = a_n(\frac{d}{dt} - r_1) \ldots (\frac{d}{dt} - r_s)(\frac{d}{dt} - r_0)^{n-s}$$

$$f(\frac{d}{dt})(y) = a_n(\frac{d}{dt} - r_1) \ldots (\frac{d}{dt} - r_s)(\frac{d}{dt} - r_0)^{n-s}(y).$$

Take $y = t^m e^{r_0 t}$, with $m \leq n-s-1$.

$$(\frac{d}{dt} - r_0)(t^m e^{r_0 t}) = m t^{m-1} e^{r_0 t}.$$

$$\therefore \quad (\frac{d}{dt} - r_0)^m (t^m e^{r_0 t}) = m! \, e^{r_0 t}.$$

$$(\frac{d}{dt} - r_0)^{m+1}(t^m e^{r_0 t}) = 0.$$

then we have: $f(\frac{d}{dt})(t^m e^{r_0 t}) = 0$

for $m \leq n-s-1$.